Bayesian analysis of level-spacing distributions for chaotic systems with broken symmetry

A. Y. Abul-Magd,* C. Dembowski,† H. L. Harney, and M. H. Simbel*
Max-Planck-Institut für Kernphysik, D-69029, Heidelberg, Germany

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Bayesian inference is applied to the nearest-neighbor and next-nearest-neighbor spacing distributions of levels of coupled superconducting microwave billiards. The weakly coupled resonators are equivalent to a quantum system with a partially broken symmetry. The coupling parameters are obtained with help from Bayes's theorem. This procedure does not require the introduction of a set of bins. The results are more accurate than those obtained from other bin-independent procedures.

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I. INTRODUCTION

Bayesian inference is the process of fitting a probability model to a set of data and summarizing the result by a distribution of the parameters of the model. The Bayesian method has found a wide application in numerous branches of science [1]. It has only recently been applied to the analysis of the statistical properties of energy spectra [2]. There, the authors considered the resonance spectra of a pair of electromagnetically coupled superconducting microwave resonators, each having the shape of a quarter of a Bunimovich stadium billiard [3]. The mean number variance—usually called the Σ^2 statistic—has been investigated in [2] for six different physical couplings. Although the six cases were intentionally chosen to be close to each other and to the two Gaussian orthogonal ensembles (GOE's) behavior, Bayesian inference yielded an accurate description of the coupling strengths.

The present paper reports a further analysis of the experiment of Ref. [3]. We consider the nearest-neighbor-spacing (NNS) and the next-to-nearest-neighbor-spacing (NNNS) distributions of levels of the coupled billiards. The spacing is denoted by s. A parameter q is inferred that specifies the coupling strength. The distribution p of s conditioned by the parameter q is the statistical model $p(s|q)$. The models for the NNS and the NNNS are discussed and defined in Secs. II and III, respectively.

Experimental distributions are usually represented by histograms and the parameters of the statistical model are determined by help of a χ^2 fit. However, the shape of a histogram depends on the choice of the bins. Even if this effect is small, it will influence the result, when small changes of the parameters have to be distinguished. This is the case in the present investigation. Furthermore, the error introduced by the choice of the bins is not assessed by the χ^2 fit. The Bayesian method directly deals with the measured spacings and not with some representation of their distribution.

Bayesian data analysis usually proceeds in three steps. First, one proposes a probability distribution for the observable $\mathbf{s}$ conditioned by the parameter q to be determined. Within the logic of statistical inference, q is called the hypothesis; the observed N spacings $\mathbf{s}=\{s_i\}$, with $i=1,\ldots,N$, are called the events. In the second step, one calculates the distribution q given the events $\mathbf{s}$. This is called the posterior distribution $P(q|\mathbf{s})$. The third step is to provide a "fit," i.e. the optimum q together with a suitable error.

In Sec. II the three-step Bayesian procedure is applied to the NNS distributions of six measured spectra. Each spectrum corresponds to a certain physical coupling of the resonators of Ref. [3]. In the experiment, the coupling is controlled by an electromagnetic connection between the resonators, see Refs. [2,3]. The measured spectra show a gradual transition from the two GOE's statistics to that of a GOE as the coupling strength increases. We assign the NNS distribution that one of the present authors has previously proposed [4] for the level statistics of systems with mixed regular-chaotic dynamics. The formula interpolates between the Wigner and Poisson distributions by tuning the probability q that the system is "locally" regular. Nevertheless, this formula provides a good approximation to the NNS distribution for spectra formed by superposing m independent level sequences [5]. In this case, the parameter $q=1-\langle f\rangle$, where $\langle f\rangle$ is the mean fractional level density of the superimposed sequences. The spectra under consideration [2] are close to this case with $m=2$ with initial level densities nearly 0.4 and 0.6. We apply this formula to describe the observed transition from two GOE's to a single GOE by varying the parameter q from nearly 0.5 to 0. In Sec. III we propose a similar formula for the NNNS distribution and use it in another Bayesian analysis of the same data aiming at the same parameter. Section IV compares the results of the present analysis with the previous Bayesian analysis of the level number variance for the same billiards [2]. The summary and conclusions are given in Sec. V.

II. ANALYSIS OF NNS DISTRIBUTIONS

The first step in Bayesian inference is to assign a conditional probability density p for the NNS's, given the parameter q. We use the expression proposed in [4] for a system of mixed regular-chaotic dynamics. This formula is derived with the help of a simple approach due to Wigner [6]. He writes the probability density p of the NNS in the form

$$p(0,s)=r(0,s)\exp\left(-\int_0^s r(0,x)\,dx\right). \tag{1}$$

*Permanent address: Faculty of Science, Zagazig University, Zagazig, Egypt.

†Permanent address: Institut für Kernphysik der Technischen Hochschule Darmstadt, D-64289 Darmstadt, Germany.

Here, $r(0,s)ds$ is the conditional probability—given a level at energy E and no level in the interval $(E,E+s)$—that the next level occurs within $(E+s,E+s+ds)$. The function $r(0,s)$ is the level-repulsion function of order 0. Later—in Sec. III—the higher order level repulsion function $r(n,s)$ will play a role. It is the conditional probability—given a level at E and n levels in the interval $(E,E+s)$—that the next level occurs within $(E+s,E+s+ds)$. Similarly, $p(n,s)$ is Wigner's density for higher order neighbors.

Note that we use Mehta's notation [7] for the function $p(0,s)$ instead of the frequently used $P(s)$ in order to have analogous notation for the high-order spacing distributions. Also we reserve capital P to the posterior distribution introduced below.

For a regular system, one takes $r(0,s)=r_P(s)=1$ to obtain the Poisson distribution. For a chaotic system described by a GOE random matrix, the choice $r(0,s)=r_W(s)=(\pi/2)s$ leads to the Wigner distribution. For a mixed system, Ref. [4] suggests that the level repulsion function is taken to be a linear superposition of r_P and r_W:

$$r(0,s)=q+\frac{\pi}{2}(1-q)s. \tag{2}$$

Here, the parameter q is the probability that the splitting of close levels is too small to be observed. Hence, q is a measure for the regular fraction of the classical phase space. Substituting Eq. (2) into Eq. (1) yields the expression

$$p(0,s)=\left[q+\frac{\pi}{2}(1-q)s\right]\exp\left[-qs-\frac{\pi}{4}(1-q)s^2\right] \tag{3}$$

for the NNS distribution. This formula provides an interpolation between the Wigner $(q=0)$ and the Poisson $(q=1)$ distributions. It has successfully been applied to analyze spacing distributions of many atomic nuclei [8].

Formula (3) provides an accurate description for the NNS distribution of a spectrum formed by superposing m independent and uncoupled GOE sequences. This is shown in Ref. [5]. Using a method elaborated by Mehta [7], an expression is obtained for the cumulative spacing distribution. Its logarithm is then expanded in powers of s keeping only the leading two terms. This approximation yields an expression for the NNS distribution of the form (3) with $q=1-\langle f\rangle$, where $\langle f\rangle=\Sigma_{i=1}^m f_i^2$ is the mean fractional level density for the superimposed sequences; the statistical weight of each sequence is given again by its fractional density. In the case of the superposition of two independent sequences of fractional densities f and $1-f$, one sets q equal to

$$q=2f(1-f). \tag{4}$$

Formula (3) then provides a good approximation to the NNS distribution. More arguments in favor of this approximation will be corroborated below.

We shall apply formula (3) also to the case when the two superposed GOE sequences are coupled to each other. By varying the parameter q from an initial value of $2f_0(1-f_0)$, with $f_0\leqslant 1/2$, to 0 we hope to be able to describe the transition from 2-GOE statistics to that of a GOE. Hence, the effect of coupling is modeled by artificially allowing one sequence to grow at the expense of the other. The GOE limit is reached when one sequence practically dominates the spectrum. We shall obtain in Sec. IV an ''empirical'' relation between the parameter q and the strength of coupling between the sequences when the coupling is described by the perturbation theory.

Unfortunately, expression (3) does not satisfy the condition

$$\int_0^\infty sP(s)ds=1 \tag{5}$$

of unit mean spacing. This deficiency is, however, immaterial, as we describe now.

One can satisfy condition (5) if one transforms s in Eq. (3) to s/s_0. This yields

$$p(0,s)=\frac{1}{s_0}\left[q+\frac{\pi}{2}(1-q)\frac{s}{s_0}\right]\exp\left[-q\frac{s}{s_0}-\frac{\pi}{4}(1-q)\left(\frac{s}{s_0}\right)^2\right]. \tag{6}$$

Here, s_0 is determined by Eq. (5) to be

$$s_0=\sqrt{1-q}\,e^{-q^2/\pi(1-q)}/\mathrm{erfc}[q/\sqrt{\pi(1-q)}] \tag{7}$$

and $\mathrm{erfc}(x)=(2/\sqrt{\pi})\int_x^\infty e^{-t^2}dt$ is the complementary error function.

We have convinced ourselves that formula (6) provides a good approximation to the NNS distributions of superposed GOE sequences by studying $m=2$ uncoupled level sequences of equal density. According to Eq. (4), the approximation is obtained by setting $q=1/2$ in Eq. (6). We have calculated the kth moment, $k=2,\ldots,10$, of the approximate and the exact NNS distributions. The exact distribution is given by Eq. (13) of Ref. [5]. In Table I, the moments of the approximate and the exact distributions are compared to each other. The agreement is good. This justifies the use of distribution (6) for uncoupled billiards. It is used to describe the NNS of the coupled billiards of Ref. [3]. In the absence of coupling, they correspond to the superposition of two GOE sequences with fractional density $f_0=0.4$. The correction introduced by allowing for s_0 to differ from unity is so small that the remainder of the analysis has been done with the simpler formula (3).

Strictly speaking, formula (3) is inappropriate in the presence of coupling between the billiards. Coupling removes the accidental level degeneracies so that $P(s)$ must go to 0 as s tends to 0. In spite of this, formula (3) has been successfully applied to the analysis of mixed systems like hydrogen atoms in strong magnetic fields [4] and atomic nuclei at low excited states [8]. It produced equally good fits to the data as the celebrated Brody formula which satisfies the condition $P(0)=0$. Thus the previous experience suggests that when the coupling is small enough, this effect influences only a small domain of s, usually much smaller than the width of the bins of the empirical histogram. The success of Eq. (3) is

TABLE I. The moments of the NNS and NNNS distributions of two independent level sequences of equal density. The exact values are given under the headings "Two GOE's." They are compared to the approximations (6) or (22). See text. The first column gives the order k of the moments.

Moment order	NNS		NNNS	
	Two GOE's	Eq. (6)	Two GOE's	Eq. (22)
2	1.4917	1.4831	4.6988	4.7202
3	2.7760	5.9231	12.466	12.442
4	6.0237	5.9231	36.462	36.020
5	14.704	14.403	115.70	112.83
6	39.492	38.587	393.74	378.38
7	114.94	112.17	1424.6	1347.5
8	358.53	349.88	5443.1	5063.4
9	1188.6	1161.3	21 844	19 975
10	4161.0	4074.7	91 668	82 375

demonstrated in Fig. 1, which indicates that this is indeed the situation for the coupled billiards under consideration.

In the experiment [3], two superconducting microwave resonators were electomagnetically coupled. Each resonator had the shape of a quarter of a Bunimovich stadium billiard and was manufactured from niobium sheets. The electromagnetic coupling was achieved by putting the two resonators on top of each other, drilling holes through the adjacent walls, and allowing one or two superconducting niobium pins to penetrate the two resonators via the holes. The coupling strength depends on the penetration depths (x_1,x_2) of the coupling pin into either resonator. The values of x_1 and x_2 in mm are used in the sequel to characterize the physical coupling of the resonators. The experiment provides sets of spacings s_i for each of the couplings (8,0),(5,3),(4,4),(8,6), and for a coupling by two pins with penetrations (8,6). This strongest coupling is labeled "two (8,6)." The number of levels observed in both resonators together was about 1500. This large number of levels is crucial for the following analysis. It is a characteristic feature of experiments with superconducting microwave cavities [9].

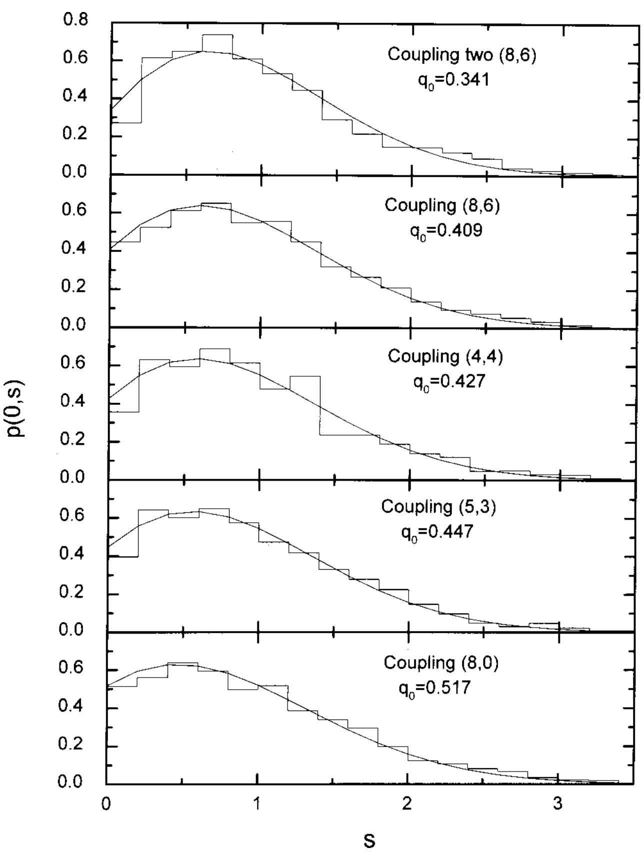

FIG. 1. The NNS distributions, calculated using Eq. (3) with the best values q_0 of the parameter q (smooth curves). For comparison, the empirical distributions obtained in the experiment [3] are given. They are represented by histograms and labeled by the physical coupling (x_1,x_2). Note, however, that the present analysis does not make use of a histogram.

We proceed to the second step of Bayesian inference from the NNS distribution. It shall express the coupling (x_1,x_2) by the parameter q. To this end, one needs the joint probability distribution $p(\mathbf{s}|q)$ of the spacings $\mathbf{s}=(s_1,s_2,\ldots,s_N)$. Taking the experimental s_i to be statistically independent, this is

$$p(\mathbf{s}|q)=\prod_{i=1}^{N} p(0,s_i). \tag{8}$$

Bayes's theorem provides the posterior distribution

$$P(q|\mathbf{s})=p(\mathbf{s}|q)\mu(q)/M(\mathbf{s}) \tag{9}$$

of the parameter q given the events s. Here, $\mu(q)$ is the so-called prior distribution and $M(\mathbf{s})=\int_0^1 p(\mathbf{s}|q)\mu(q)dq$ is the normalization. We use Jeffreys's rule [10] to find the prior distribution:

$$\mu(q)=\left|\int p(\mathbf{s}|q)[\partial \ln p(\mathbf{s}|q)/\partial q]^2 d\mathbf{s}\right|^{1/2}. \tag{10}$$

It ensures—at least approximately—unbiased inference [11] of q. We evaluate this integral numerically. It diverges at $q=0$, where it is proportional to $(-\ln q)^{1/2}$. Otherwise, $\mu(q)$ is a very slowly varying function. It decreases from the value

TABLE II. The best values and errors of the parameter q obtained via Bayesian analysis from the NNS and NNNS distributions of levels of two coupled billiards. The first column gives the physical coupling (x_1, x_2) explained in the text. The last two columns are the determination of q from the previous analysis of level number variance Σ^2 [2] and the variance of the NNS distribution as explained in Sec. V, respectively.

Coupling	NNS	NNNS	Σ^2	Moments
(8,0)	0.52 ± 0.03	0.50 ± 0.03	0.48 ± 0.02	0.51 ± 009
(5,3)	0.45 ± 0.03	0.44 ± 0.03	0.42 ± 0.02	0.45 ± 0.10
(4,4)	0.43 ± 0.03	0.43 ± 0.03	0.39 ± 0.02	$(0.43\pm0.10$
(8,6)	0.41 ± 0.03	0.40 ± 0.03	0.34 ± 0.02	0.39 ± 0.10
Two (8,6)	0.34 ± 0.04	0.39 ± 0.03		0.37 ± 0.10

of 0.5049 at $q=0.1$ to the minimum of 0.3952 at $q=0.5$ and then monotonically increases to become equal to 0.5717 at $q=0.9$. As a consequence, $\mu(q)$ can be considered constant and equal to $\mu(q_0)$ in the present analysis, where q_0 is not far from the value of 0.5. This simplifies the Bayesian analysis because $\mu(q)$ drops out of Eq. (9). Note, however, that this only seems to make the analysis independent of μ. It is the present parametrization that renders the posterior distribution nearly independent of μ. A reparametrization of the problem, i.e., the transition from q to a monotonic function of q, can alter this situation.

To numerically evaluate the posterior distribution, we have expressed Eq. (8) in the form

$$p(\mathbf{s}|q)=e^{-N\phi(q)}, \tag{11}$$

where

$$\phi(q)=q\langle s\rangle+\frac{\pi}{4}(1-q)\langle s^2\rangle - \left\langle \ln\left[q+\frac{\pi}{2}(1-q)s\right]\right\rangle. \tag{12}$$

Here, the notation

$$\langle x\rangle=\frac{1}{N}\sum_{i=1}^{N} x_i \tag{13}$$

has been used. For each of the six couplings, we found the function $\phi(q)$ to have a single minimum at, say, $q=q_0$. One can therefore represent the numerical results if one parametrizes ϕ as

$$\phi(q)=A+B(q-q_0)^2+C(q-q_0)^3. \tag{14}$$

The values of the coefficients (A,B,C) for the above-mentioned six couplings are $(0.920,0.392,-0.134)$, $(0.898,0.356,-0.057)$, $(0.893,0.328,-0.020)$, $(0.891,0.345,-0.038)$, $(0.883,0.360,-0.054)$, and $(0.873,\ 0.268,\ 0.034)$, respectively. Substituting Eqs. (11) and (14) into Eq. (9), we obtain

$$P(q|\mathbf{s})=\alpha e^{-(q-q_0)^2/2\sigma_0^2}, \tag{15}$$

where $\alpha=\mu(q)\exp[-AN-CN(q-q_0)^3]/M(\mathbf{s})$ and $\sigma_0=1/\sqrt{2BN}$. In the present analysis, the number of spacings for each coupling is so large ($N\geqslant1500$) that a Gaussian function (15) very well describes the q dependence of P, which means that the third order term in Eq. (14) is immaterial. It has been introduced to convince ourselves of the fact that it is small. Indeed, in the six cases under consideration, the posterior distribution nearly vanishes except in the range of the order of $|q-q_0|\leqslant\sigma_0\simeq0.03$. In this range, the function $\mu(q)$ is nearly constant and the quantity $CN|q-q_0|^3\leqslant0.002$, so that α can be regarded constant. Its value is

$$\alpha=1/\sqrt{2\pi}\sigma_0. \tag{16}$$

Therefore, the parameter distributions obtained in this analysis are Gaussians. Each one is characterized by a mean value q_0 and a variance σ_0. Table II gives the values of $q_0\pm\sigma_0$ obtained for each physical coupling. Figure 1 shows a comparison of the NNS distributions calculated by Eq. (3) using the most-probable values q_0 of the parameter q, together with the empirical distributions. The empirical distributions are given by histograms. We emphasize, however, that the Bayesian analysis does not make use of the histogram.

The Gaussian approximation [Eqs. (15) and (16)] can only be obtained if the dependence of the prior distribution $\mu(q)$ on the parameter q can be ignored. This excludes cases for which the parameter q is expected to be close to zero. Bayesian analysis of the Σ^2 statistic for the same data carried out in [2] uses the model for symmetry breaking by Leitner *et al.* [12]. The analysis shows that the posterior distribution is different from a Gaussian when the symmetry-breaking strength is expected to be zero. The present results together with those of Ref. [2] indicate that the posterior distributions obtained in Bayesian analysis are well represented by Gaussian functions except if the true value of q is close to the ends of the domain where q is defined. The meaning of "at the end" depends, however, on the amount N of events that one has collected: The larger N is, the closer to the "end" can the true value be for the posterior to be Gaussian.

III. ANALYSIS OF NNNS DISTRIBUTIONS

As in the preceding section, we start by assigning a probability density to the spacing between a given level and the level next to its nearest neighbor. This density will parametrically depend on the coupling parameter q. Reference [13] reports a number of models for the calculation of the NNNS distributions; but the expressions are quite cumbersome. We want to have the simplest formula that has as much in common as possible with Eq. (6) of the preceding section. For this purpose we start from a generalization of Wigner's argument that led to Eq. (1). The generalization has been proposed by Engel *et al.* [14]. It is the recursion relation

$$p(n,s)=r(n,s)\int_0^s p(n-1,x) \times\exp\left(-\int_x^s r(n,y)dy\right)dx \tag{17}$$

for the nth order level-spacing distribution. The nth order level repulsion function $r(n,s)ds$ has been explained in connection with Eq. (1). Equation (17) was used in [15] to propose a generalization of the Wigner surmise for the higher-order level-spacing distribution of chaotic systems. In Ref. [15] it was found that Eq. (6) yields wrong results for large values of s when $r(0,s)$ is taken to be proportional to s^{β} if $\beta>1$. It was therefore assumed in Ref. [15] that Eq. (17) is correct only at small values of s. This yields within the leading order in s

$$p(n,s)\propto r(n,s)\int_0^s p(n-1,x)dx \quad \text{for } s\ll 1. \tag{18}$$

An expression for the nth order level-spacing distribution of a system described by the GOE that is valid in the whole domain of s is obtained by multiplying expression (18) with a Gaussian function of s. Moreover, it was assumed in [15] that the occurrence of consecutive levels in a chaotic system is an uncorrelated random process. This allows us to express the probability $r(n,s)$ of occurrence of the $(n+1)$th level at a distance s in terms of the probability $r(0,s)$ so that

$$r(n,s)=[r(0,s)]^{n+1}. \tag{19}$$

These assumptions led in [15] to expressions for the nth order level-spacing distributions, with $n=0,\dots,7$, for the Gaussian orthogonal, unitary, and symplectic ensembles. They almost perfectly agreed with the exact results reported in Mehta's book [7]. In particular, the NNNS distribution obtained in [15] for a system described by a GOE is given by

$$p_W(1,s)=a_W s^4\exp\left(-\frac{16}{9\pi}s^2\right), \tag{20}$$

where $a_W=2(16/9\pi)^3$.

We can now obtain the NNNS distribution of levels of a mixed system. Equations (18) and (19) together with the ansatz (2) yield

$$p(1,s)\propto\left[q+\frac{\pi}{2}(1-q)s\right]^2\left[qs+\frac{\pi}{4}(1-q)s^2\right] \quad \text{for } s\ll 1. \tag{21}$$

To obtain an expression valid for all s, we multiply the right-hand side of Eq. (21) by an exponential whose argument is again a linear superposition of r_P and r_W as in Eq. (3). In analogy with Eq. (6), the variable s is transformed to s/s_1 in order to satisfy the condition that the expectation value of the s is 2. All this finally yields

$$p(1,s)=\frac{b}{s_1}\left[q+\frac{\pi}{2}(1-q)\frac{s}{s_1}\right]^2 \times\left[q\frac{s}{s_1}+\frac{\pi}{4}(1-q)\left(\frac{s}{s_1}\right)^2\right] \times\exp\left[-q\frac{s}{s_1}-\frac{16}{9\pi}(1-q)\left(\frac{s}{s_1}\right)^2\right]. \tag{22}$$

The quantities b and s_1 are determined from the conditions

$$\int_0^\infty p(1,s)ds=1 \quad \text{and} \quad \int_0^\infty sp(1,s)ds=2. \tag{23}$$

Formula (22) interpolates between the Poisson distribution for the NNNS for $q=1$ and the generalized Wigner distribution (20) for $q=0$. If $q=0.5$, it provides a good approximation to the NNNS distributions obtained in [13] for the superposition of two independent GOE sequences having the same density.

To show this, we calculated the kth moment, $k=2,\dots,10$, of the NNNS distribution for a superposition of two GOE level sequences of equal density. The exact expression is given by Eq. (25) of Ref. [13]. In Table I its moments are compared to the moments of the approximation (22). The agreement is good except for the highest moments, which anyway measure the behavior of the distribution at spacings where the probability density is small.

As in the case of the NNS distribution, we set $s_1=1$. The normalization factor is approximated by

$$b=\frac{16}{\pi^3}a_W\exp(\nu), \tag{24}$$

$$\nu=1.0388q+0.49093q^2-0.56813q^3+0.71021q^4, \tag{25}$$

where the preexponential factor is chosen such that Eq. (22) yields the GOE result (20) when $q=0$.

We proceed to the next step of the Bayesian analysis and define the joint distribution p_1 of s, given the parameter q, by

$$p_1(\mathbf{s}|q)=\prod_{i=1}^N p(1,s_i). \tag{26}$$

As in Sec. II, the measured spacings s_i are considered as statistically independent events.

In analogy with Eq. (10), the prior distribution is

$$\mu_1(q)=\left|\int p_1(\mathbf{s}|q)[\partial\ln p_1(\mathbf{s}|q)/\partial q]^2 d\mathbf{s}\right|^{1/2}. \tag{27}$$

It is easy to show by numerical calculation that $\mu_1(q)$ is a slowly varying function for $0.1<q<0.9$ and can be considered constant in the present context—very much as in Sec. II. Similarly to Eq. (11), the joint probability density of the NNNS is represented as

$$p_1(\mathbf{s}|q)=\left(\frac{16}{\pi^3}a_W\right)^N e^{-N\phi_1(q)}, \tag{28}$$

where

$$\phi_1(q)=q\langle s\rangle+\frac{16}{9\pi}(1-q)\langle s^2\rangle -\nu-\left\langle 2\ln\left[q+\frac{\pi}{2}(1-q)s\right] +\ln\left[qs+\frac{\pi}{4}(1-q)s^2\right]\right\rangle. \tag{29}$$

We numerically evaluate the function $\phi_1(q)$ and again represent the results by the parametrization $\phi_1(q)=A_1+B_1(q-q_1)^2+C_1(q-q_1)^3$. The values of the coefficients (A_1,B_1,C_1) for the six physical couplings are $(-0.424,0.520,0.244)$, $(-0.461,0.469,0.247)$, $(-0.457,0.443,0.121)$, $(-0.473,0.435,0.083)$, $(-0.456,0.446,0.116)$, and $(-0.491,0.417,0.393)$, respectively. The posterior distribution for the NNNS is defined in analogy with Eq. (9). With the help of Eqs. (27) and (28), one obtains

$$P_1(q|\mathbf{s})=\alpha_1 e^{-(q-q_1)^2/2\sigma_1^2}. \tag{30}$$

Here

$$\alpha_1=\mu_1(q)\left(\frac{16}{\pi^3}a_W\right)^N \times\exp[-A_1N-C_1N(q-q_1)^3]/M_1(\mathbf{s}), \tag{31}$$

$$M_1(\mathbf{s})=\int_0^1 p_1(\mathbf{s}|q)\mu_1(q)dq, \tag{32}$$

and

$$\sigma_1=1/\sqrt{2B_1N}. \tag{33}$$

Since σ_1 is small, the dependence of α_1 on q is negligible and therefore P_1 is Gaussian and α_1 has the value $\alpha_1=1/\sqrt{2\pi}\sigma_1$.

In Table II, the values of q_1 and σ_1 are given in the form of $q_1\pm\sigma_1$. Figure 2 shows the NNNS distributions $p(1,s)$, calculated by Eq. (22) with q set equal to q_1, together with the empirical distributions. The latter ones are represented by histograms, although the Bayesian analysis does not refer to a histogram. In some cases, the theoretical curves are slightly shifted towards larger spacing. The reason for the shift is that we have set the quantity s_1 in Eq. (22) equal to unity, which leads to a somewhat too large mean spacing. We have done so in order to keep the analysis as simple as possible.

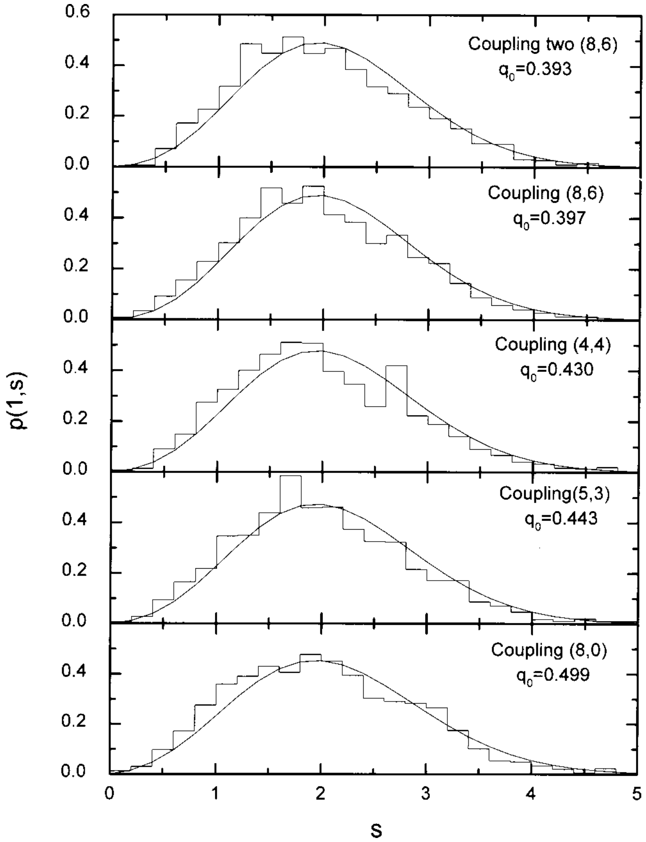

FIG. 2. The NNNS distributions, calculated using Eq. (22) with the best values q_0 of the parameter q (smooth curves). For comparison, the empirical distributions obtained in the experiment [3] are given. See the caption of Fig. 1.

IV. COMPARISON WITH AN ANALYSIS OF THE NUMBER VARIANCE

A Bayesian analysis of the level number variance Σ^2 for the coupled billiards under consideration are given in [2]. The authors of this paper use a formalism developed by Leitner [12] by applying the perturbation theory to the coupling between two chaotic billiards. Leitner considers a Hamiltonian consisting of a sum of a block-diagonal matrix, representing the case when the symmetry is conserved, and a perturbation responsible for symmetry breaking. When the symmetry has two eigenvalues, the Hamiltonian takes the form

$$H=\begin{pmatrix} H_1 & 0\\ 0 & H_2\end{pmatrix}+\varepsilon\begin{pmatrix} 0 & V\\ V^\dagger & 0\end{pmatrix}, \tag{34}$$

where H_1, H_2, and V are GOE matrices having same rms value v so that $\varepsilon=1$ makes H as a whole to be a GOE matrix. In the simplest case of two sequences of equal fractional density, the NNS distribution P_{L}, where the subscript L stands for Leitner, is given by [12]

$$P_{\mathrm{L}}(s,\Lambda)=c_N c_D\left[\sqrt{\frac{\pi}{32\Lambda}}I_0\left(\frac{c_D^2 s^2}{16\Lambda}\right)se^{-c_D^2 s^2(1+2\pi\Lambda)/16\Lambda}+\frac{\pi}{8}se^{-\pi c_D^2 s^2/16}\,\mathrm{erfc}\left(\frac{\sqrt{\pi}}{8}c_D s\right)\right], \tag{35}$$

where $I_0(x)$ is a Bessel function, $\Lambda=(\varepsilon/D)^2$, and D is the mean level spacing. The coefficients c_N and c_D are set so that P_{L} is normalized and $\langle s\rangle=1$, respectively. The normalization integral can be solved analytically leading to the following expression for c_N:

$$c_N=\frac{c_D}{1-1/\sqrt{5}+1/\sqrt{2(1+\pi\Lambda)}}, \tag{36}$$

$$c_D=\frac{2\left[\{-4+5\pi-10\arctan(1/2)\}/5\pi+\sqrt{2}\pi\Lambda(1+2\pi\Lambda)^{-3/2}F\left(\frac{3}{4},\frac{5}{4};1;\frac{1}{(1+\pi\Lambda)^2}\right)\right]}{1-1/\sqrt{5}+1/\sqrt{2(1+\pi\Lambda)}}, \tag{37}$$

where $F(a,b;c;z)$ is the hypergeometric function [17]. We have not succeeded in expressing the hypergeometric function in the expression for c_D in terms of elementary functions. Thus, each term of the distribution (35) contains a special function of a special function. This makes using Eq. (35) in a Bayesian analysis of the NNS distribution much more complicated than using Eq. (3). Moreover, its derivation does not allow a simple generalization to obtain higher-order level-spacing distributions.

Leitner's approach leads, on the other hand, to the following simple expression for a level-number variance in a spectral interval of length r:

$$\Sigma_{\mathrm{L}}^2(r,\alpha)=\Sigma_{\mathrm{GOE}}^2(r)+\frac{1}{\pi^2}\ln(1+\alpha^2 r^2), \tag{38}$$

where $\alpha=\pi/2(\tau+\pi^2\Lambda)$ and τ is determined by the requirement that $\Sigma_{\mathrm{L}}^2(r,\pi/2\tau)$ coincides with that of two independent GOE sequences, while Σ_{GOE}^2 is the level-number variance for a GOE [16]. This formula has been successfully used in [2] to obtain the coupling parameter Λ for the billiards presently under consideration by Bayesian inference.

We would like to compare the results of the earlier analysis of the number variance reported in Ref. [2] with those of the present analysis of the NNS and NNNS distributions. For this purpose, we need a relation between the parameter q of Eqs. (3) and (22) and the parameter α (or Λ, equivalently) of Eq. (38). We recall that Eqs. (3) and (22) are used as a representation for the corresponding formulas for the spacing distribution of two independent sequences of fractional densities f and $1-f$. In the same spirit, we represent the number variance statistic (38) by a sum of two GOE terms with the corresponding level densities:

$$\Sigma_{\mathrm{L}}^2(r,\alpha)\cong\Sigma_{\mathrm{GOE}}^2(fr)+\Sigma_{\mathrm{GOE}}^2(\{1-f\}r). \tag{39}$$

We find the relation between the coupling parameter α (or effective perturbation strength Λ) and the fractional density f by minimizing the quantity

$$\chi^2=\frac{4}{N}\sum_{k=1}^{N}\left[\frac{\Sigma_{\mathrm{GOE}}^2(fr)+\Sigma_{\mathrm{GOE}}^2(\{1-f\}r)-\Sigma_{\mathrm{L}}^2(r,\alpha)}{\Sigma_{\mathrm{GOE}}^2(fr)+\Sigma_{\mathrm{GOE}}^2(\{1-f\}r)+\Sigma_{\mathrm{L}}^2(r,\alpha)}\right]^2 \tag{40}$$

with respect to α for different values of f. The results are given in Table III for $N=20$. The small values of χ^2 that minimize Eq. (40) justify the approximation (39). The χ^2-fit values of α almost linearly depend on $f(1-f)$ for not-too-small values of f. We may conclude that Eq. (38) holds when

$$\alpha=c_1 f(1-f) \tag{41}$$

with $c_1=8.94\pm0.04$ for $0.2<f<0.5$. Comparing Eqs. (4) and (41) we obtain the following relation between the parameters q of Eqs. (3) and (22) and the parameter α or Λ of Leitner's formulas:

$$q=c_2\alpha=\frac{c_3}{\tau+\pi^2\Lambda}, \tag{42}$$

where $c_2=2/c_1$ and $c_3=\pi/c_1$. This relation allows the comparison with the previous analysis of the number variance of levels of the coupled billiards under consideration [2]. Table II gives the values of q calculated by Eq. (42) using the values of Λ reported in [2] from the analysis of Σ^2. These values are, on the average, 10% smaller than the corresponding values obtained in the present analysis of the NNS and NNNS distributions. This difference lies within the statistical errors.

V. SUMMARY AND CONCLUSIONS

The statistical analyses of level-spacing distributions are among the most popular methods to study systems with mixed regular-chaotic classical dynamics. The distributions of the data are usually represented in the form of a histogram. A formula that interpolates between the distributions for a regular and a chaotic system by tuning a parameter q is adjusted to the empirical histogram. The choice of the bins

TABLE III. Least square fit of Leitner's formula for the number variance Σ^2 of two chaotic systems interacting with strength expressed by the parameter α to the corresponding formula for two independent level sequences with fractional densities f and $1-f$.

f	α	χ^2
0.50	2.226	6.89×10^{-7}
0.45	2.203	7.56×10^{-7}
0.40	2.138	9.41×10^{-7}
0.35	2.029	1.38×10^{-6}
0.30	1.877	2.39×10^{-6}
0.25	1.682	4.94×10^{-6}
0.20	1.443	1.19×10^{-5}
0.15	1.161	3.08×10^{-5}
0.10	0.831	7.75×10^{-5}
0.05	0.451	1.70×10^{-4}

will influence the value of the parameter that results from the fit. The present paper suggests using the method of Bayesian inference for the evaluation of the parameter. This analysis deals with the spacings directly, and bypasses the introduction of a histogram.

We have demonstrated the application of the Bayesian analysis by the example of the spacing distributions from two weakly coupled superconducting microwave billiards. For the NNS distribution, a formula has been used that was previously proposed by one of the present authors. This formula sets a reasonable approximation for the superposition of independent sequences of chaotic spectra. We use it to describe the transition from the two GOE statistics to that of a single GOE. For the NNNS distribution, an analogous formula has been derived in the present paper. Both distributions depend on one and the same parameter q that measures the strength of the coupling between both microwave resonators.

In the two cases, the prior distributions only weakly depend on the parameter q except for the vicinity of the ends 0 and 1 of its range. The spectra under consideration correspond to values of $q \lesssim 0.5$. The resulting posterior distributions of q for both the NNS and NNNS cases, are narrow Gaussians. Their mean values and variances are given in Table II. The accuracy obtained for the parameter q is in the range of 6–8 %. Within this error, the values of q obtained from both observables, the NNS and the NNNS, are in agreement—as they should be. They also agree with the outcome of a previous Bayesian analysis of the number variance of the levels of the same billiards using a perturbative formalism developed by Leitner. This comparison is done by deducing a numerical relation between the parameter q and the perturbation strength.

The accuracy can be compared with that of another way to obtain q—a way that shares with Bayesian inference the feature to bypass the introduction of a histogram. One can calculate the mean square value of the NNS distribution $p(0,s)$ as a function of q. From the experimental mean square value together with its error one can then infer q. The result is given in Table II under the heading "Moments." In this case, the accuracy is in the range of 25–30 %. There is no theorem available that would prove the Bayesian error interval to be smaller than that obtained with the help of any statistic—such as the second moment used above. However, from the experience that we have gained in the course of the present study, this may be so. If this is true, however, then neither the NNS nor the NNNS provides the most precise determination of the coupling parameter q since both of them are a statistic, i.e. a function of the primary data set, which is the spectrum as a whole. One would therefore like to base the Bayesian analysis on the distribution of the whole spectrum conditioned by q. This model, however, is not available.

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